Helioseismic study of the sun variability during the activity cycle

1 Introduction

We study global changes in the solar eigenmode frequencies over the sun's activity cycle. The purpose is to interpret of the helioseismic data in terms of variations magnetic and velocity field. Our tool, like in all previous studies of the same problem is a variational principle for oscillation frequencies.

In part, our work expands the method adopted by Goldreich et al.(1991, GETAL) in treating the effect of small scale magnetic field on nonradial modes and the nonspherical distribution of the averaged field. Although it is true that radial modes may adequately represent all nonradial p-modes, if the magnetic field effects are confined to the outermost layers, it is not true for f-modes. More importantly, we consider nonspherical field distribution, which allows us to interpret the observed evolution of the even-a coefficients describing aspherical changes in the sun.

We consider also the effect of the global toroidal field expected near the base of the convective envelope. Present work, which represents an improvment of earlier ones (Dziembowski & Goode, 1991, and Gough & Thomson ,1991), is aimed at determinig a stringent limit on the size of such a field.

Furthermore, we study effects of large and steady flow fields, as well as a small scale velocity random velocity field.

2 Variational expresion for oscillation frequencies

There are two ways of deriving the variational expression for oscillation frequencies. One always relies on adiabatic approximation and this adopted throughout our study. One approach begins with the linearized equations of fluid motion about a steady configuration (see e.g Lynden-Bell & Ostriker, 1967, L&O) the other with the Hamilton's principle (see e.g. GETAL; Dewar, 1970). Here we use the form given by L&O with some simplification and adding the contribution from magnetic field as calculated explicitely by Dewar. The L&O form is valid for strictly steady velocity fields. However, we make certain simplification, which will be explained later, to make it appplicable to statistically steady fields. With this, we write

$$\omega^2 I = -\omega C + D \tag{1}$$

where

$$I = \int d^3\boldsymbol{x}\rho|\boldsymbol{\xi}|^2 \tag{2}$$

$$C = 2\mathrm{i} \int d^3\boldsymbol{x}\rho\boldsymbol{\xi}^* \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi}. \tag{3}$$

The eigenvectors, in a spherically-symmetric and time independent model of the sun are expressed in the following standard form

$$\boldsymbol{\xi} = r[y(r)\boldsymbol{e}_r + z(r)\boldsymbol{\nabla}_H]Y_\ell^m(\theta, \phi)\exp(-\mathrm{i}\omega t). \tag{4}$$

We adopt approximations regarding the eigenfunctions. In addition to adiabaticity we assume the Cowling approximation, which is well justified in our application to solar oscillations. Further, we will make use of the fact that the oscillations are either of high degree or order, which means that

$$|\boldsymbol{\xi}| \ll \mathrm{Max}(r|\boldsymbol{\xi}_{;r}|, \ell|\boldsymbol{\xi}|).$$

Equivalent approximations were also made by GETAL.

Like L&O, we split various contribution to D,

$$D = D_p + D_g + D_v + D_M. \tag{5}$$

The pressure term,

$$D_p = \int d^3\boldsymbol{x}p[\Xi + (\Gamma - 1)|\mathrm{div}\boldsymbol{\xi}|^2], \tag{6}$$

is the same as in L&O. The quantity Ξ is a completely contracted double dyadic product, $\boldsymbol{\nabla\xi}^* : \boldsymbol{\nabla\xi}$. Adopting the standard summation convention, we have

$$\Xi = \xi^*_{j;k}\xi_{k;j} = (\xi^*_j\xi_{k;j} - \xi^*_k\mathrm{div}\boldsymbol{\xi})_{;k} + |\mathrm{div}\boldsymbol{\xi}|^2,$$

where subscript ; denotes covariant derivatives. However, with our approximation regarding $\boldsymbol{\xi}$, contributions from the terms involving the Christoffel symbols is negligible and the derivatives may be regarded as component derivatives. In terms of the radial eigenfunctions, y and z, with the adopted approximations we have

$$\begin{aligned}\Xi \ \approx\ & [r(ryy_{;r})_{-}y\lambda)_{;r} + \lambda^2]|Y_\ell^m|^2 + r(yz)_{;r}|\boldsymbol{\nabla}_H Y_\ell^m|^2 \\ & +[0.5(yrz_{;r} - z\lambda)|Y_\ell^m|^2_{;\theta} + z^2\boldsymbol{\nabla}_H Y_\ell^{m*} \cdot \boldsymbol{\nabla}_H Y^m_{\ell;\theta}]_{;\theta} + [...]_{;\phi},\end{aligned}$$

where

$$\lambda = y\frac{gr}{c^2} - z\frac{\omega^2 r^2}{c^2}$$

is radial eigenfunction corresponding to div$\boldsymbol{\xi}$, g is the local gravity, and c is the speed of sound. The last term in Ξ is obtained from the previous one by replacing θ with ϕ. Further, in the adopted approximation we have

$$ry_{;r} = \lambda + \Lambda z, \tag{7}$$

where $\Lambda = \ell(\ell+1)$ and

$$rz_{;r} = y - \lambda\left(\frac{N}{\omega}\right)^2 \frac{c^2}{gr}. \quad (8)$$

The term containing the Brunt-Väisälä frequency, N, is of the same order as the first one only for p-modes and only in the outermost layers. However, the whole contribution from the term involving $rz_{;r}$ is then small. Hence, we will ignore the term.

$$\Xi = \lambda^2|Y_\ell^m|^2 + (y^2 + \lambda z + \Lambda z^2)(\Lambda|Y_\ell^m|^2 + \boldsymbol{\nabla}_H Y_\ell^m|^2) + [...]_{;\theta} + [...]_{;\phi}, \quad (9)$$

The explicit expressions for the last two terms will not be needed.

The gravity term simplifies to

$$D_g = -2\int d^3\boldsymbol{x}\rho\frac{g}{r}|\xi_r|^2 \quad (10)$$

after using the Cowling approximation. The velocity term ,

$$D_v = -\int d^3\boldsymbol{x}\rho|(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}|^2, \quad (11)$$

is the same as in L&O, where it was derived for a steady field velocity field. We will use the same form in our application to a statistically steady turbulent field. The expression for the magnetic term, which is taken from Dewar (1970), is

$$D_M = \frac{1}{4\pi}\int d^3\boldsymbol{x}[|(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}|^2 - 2\mathrm{div}\boldsymbol{\xi}^*\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\xi} + \frac{1}{2}|\boldsymbol{B}|^2(\Xi + |\mathrm{div}\boldsymbol{\xi}|^2)] \quad (12)$$

3 Time and longitude dependent perturbation in the sun's structure

Here, we include dynamical effects of the magnetic $\boldsymbol{B}$ and and those of velocity fields $\boldsymbol{v}$. We write mechanical equilibrium condition in the presence of perturbing force $\boldsymbol{F}$ in the following form,

$$\nabla p + \rho g\boldsymbol{e}_r = \boldsymbol{F}, \quad (13)$$

where

$$\boldsymbol{F} \equiv -\left(\frac{\partial\mathcal{V}}{\partial r}\boldsymbol{e}_r + \frac{\partial\mathcal{H}}{\partial\theta}\frac{\boldsymbol{e}_\theta}{r}\right) = -\frac{1}{4\pi}\left[\frac{1}{2}\nabla\boldsymbol{B}^2 - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\right] - \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \quad (14)$$

We neglected perturbation of the gravitational potential.

Eq.(13) implies

$$p = -\mathcal{H} + h(r) \tag{15}$$

and

$$\rho = \frac{1}{g}\frac{\partial}{\partial r}(\mathcal{H} - \mathcal{V} - h). \tag{16}$$

The quantity $h(r)$ may only be determined with the thermal equilibrium condition.

From now on, we treat $\boldsymbol{F}$ as a small perturbing force. Primed letters denote Eulerian perturbations of the respective structure parameters , letters preceded by δ denote Lagrangian perturbations, and letters without such symbols denote unperturbed parameters. We use standard relation

$$\delta f = f' + \frac{df}{dr}\delta r$$

and we adopt $\delta M_r = 0$, both for spherical and aspherical perturbations.

3.1 Spherically symmetric changes

If, following GETAL, we use a plane-parallel approximation, we obtain from Eqs.(15) and (16)

$$\delta p = -\delta\mathcal{V} \tag{17}$$

The general expression is

$$\delta p = -\delta\mathcal{V} - 4\int_r^R dr\frac{\delta r}{r}g\rho. \tag{17a}$$

Mass conservation implies

$$\frac{d\delta r}{dr} = -\frac{\delta\rho}{\rho} - 2\frac{\delta r}{r}, \tag{18a}$$

which with GETAL's approximation simplifies to

$$\frac{d\delta r}{dr} = -\frac{\delta\rho}{\rho}. \tag{18}$$

With the help of the EOS we may now express all the Lagrangian variations of all parameters in terms of $\delta\mathcal{V}$, and δT, e.g.

$$\frac{\delta\rho}{\rho} = -\frac{1}{\chi_\rho}\left(\frac{\delta\mathcal{V}}{p} + \chi_T\frac{\delta T}{T}\right) \tag{19}$$

or the entropy per mass variation, δS, in place of δT, e.g.

$$\frac{\delta\rho}{\rho} = \frac{1}{\Gamma}\frac{\delta\mathcal{V}}{p} - \frac{\chi_T}{\chi_\rho}\frac{\delta S}{c_p}. \tag{20}$$

Here we used standard notation in astrophysics, e.g. χ's denote derivatives of $\log p$ with respect of the $\log\rho$ and $\log T$. We do not put, however, subscript 1 at Γ to avoid confusion with the expansion coefficient.

3.2 Asphericity

We express Eulerian perturbations p' and ρ' in the form of the even-order Legendre polynomial series. The subscript k denotes the r-dependent coefficient at P_{2k}. From Eqs.(15) and (16) we get for $k > 0$,

$$p_k = -\mathcal{H}_k, \qquad \text{and} \qquad \rho_k = \frac{1}{g}\frac{d}{dr}(\mathcal{H}_k - \mathcal{V}_k). \tag{21}$$

Corresponding Lagrangian perturbations (with GETAL approximation assumed) are

$$\delta_k p = -\mathcal{V}_k = p_k + \delta_k r\frac{dp}{dr}, \qquad \text{and} \qquad \delta_k\rho = \rho_k + \delta_k r\frac{d\rho}{dr} \tag{22}$$

The relation between p_k and $\delta_k p$ implies

$$\delta_k r = \frac{\mathcal{V}_k - \mathcal{H}_k}{g\rho}. \tag{23}$$

With these expressions, the temperature perturbation is found as

$$\frac{\delta_k T}{T} = -\frac{1}{p\chi_T}\left[\mathcal{H}_k + \left(\chi_\rho\frac{d}{d\ln p} + \chi_T\frac{d\ln T}{d\ln p}\right)(\mathcal{V}_k - \mathcal{H}_k)\right]. \tag{24}$$

3.3 Distortion due to steady flows

Rotation and meridional circulation are flows in the Sun which may be regarded steady. Although solar rotation is not uniform, such an approximation is sufficient for our aim here. With this approximation we have for centrifugal force

$$\boldsymbol{F} = -\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \frac{\Omega^2\rho}{3}\{(2r\mathbf{e}_r - \boldsymbol{\nabla}[r^2P_2(\cos\theta)]\},$$

which is, of course, the centrifugal force. Thus, we have

$$\frac{d\mathcal{V}_0}{dr} = -\frac{2}{3}\rho\Omega^2 r, \quad \frac{d\mathcal{V}_1}{dr} = \frac{2}{3}\rho\Omega^2 r, \quad \mathcal{H}_1 = \frac{1}{3}\rho\Omega^2 r^2 \tag{25}$$

To estimate the inertia arising from meridional circulation, we assume a P_2 angular dependence for the radial component of the velocity and make use of the continuity equation, $\mathrm{div}(\boldsymbol{v}\rho) = 0$. Thus, we may express two velocity components with the scalar function ,$f(r)$, as follows.

$$v_r = f(r)P_2(\cos\theta), \qquad v_\theta = h(r)\frac{dP_2}{d\theta}, \qquad h \equiv \frac{1}{6r\rho}\frac{d(fr^2\rho)}{dr}.$$

In the outermost layers, both f and ρ are rapidly varying functions of r, thus there is much larger value for the horizontal component than the vertical component of the velocity. It seems unlikely that we have $f\rho \approx$ constant everywhere there. In evaluating perturbing force we keep only the dominant term and we obtain

$$\frac{\partial \mathcal{V}}{\partial r} = -\frac{\rho h^2}{r}\left(\frac{dP_2}{d\theta}\right)^2, \qquad \mathcal{H} = \frac{\rho}{2}\left[fr\frac{dh}{dr}P_2^2 + h^2\left(\frac{dP_2}{d\theta}\right)^2\right],$$

or

$$\frac{d\mathcal{V}_0}{dr} = -\frac{6}{5}\frac{\rho h^2}{r}, \quad \frac{d\mathcal{V}_1}{dr} = -\frac{6}{7}\frac{\rho h^2}{r}, \quad \frac{d\mathcal{V}_2}{dr} = \frac{72}{35}\frac{\rho h^2}{r} \tag{26}$$

and

$$\mathcal{H}_1 = \frac{\rho}{7}\left(fr\frac{dh}{dr} + 3h^2\right), \quad \mathcal{H}_2 = \frac{9\rho}{35}\left(fr\frac{dh}{dr} - 4h^2\right) \tag{27}$$

3.4 Toroidal magnetic field

The field $\boldsymbol{B} = B_{t,2}(r)\sin\theta\cos\theta\boldsymbol{e}_\phi$ gives rise to he Lorentz force given by

$$\boldsymbol{F} = -\frac{\boldsymbol{\nabla}(B_{t,2}^2 r^2\sin^2\theta)}{8\pi r^2\sin^2\theta} \approx -\frac{\boldsymbol{\nabla}(B_{t,2}^2\sin^2\theta)}{8\pi\sin^2\theta}.$$

The last approximation is valid for the field confined to a narrow layer, which we will assume here. So that we have

$$\mathcal{V} = \frac{B_{t,2}^2}{8\pi} \quad \text{and} \quad \mathcal{H} = \frac{1}{8\pi}\int\frac{d\theta}{\sin^2\theta}\frac{\partial(B_{t,2}^2\sin^2\theta)}{\partial\theta}.$$

We now put $B_t(r,\theta)$ in the form of the following series

$$B_t(r,\theta) = \sum_j \sqrt{\frac{2j+1}{2j(j+1)}}B_{t,j}(r)\frac{dP_j(\cos\theta)}{d\theta}. \tag{28}$$

Note that with this representation $B_{t,j}$ represent the surface averaged intensity of field component at distance r from the center. Considering only two first component, we get the following non-zero components in the Legendre polynomials expansion of $\mathcal{V}$

$$\mathcal{V}_0 = \frac{B_{t,1}^2 + B_{t,2}^2}{16\pi}, \quad \mathcal{V}_1 = \frac{1}{16\pi}\left(-B_{t,1}^2 + \frac{5}{7}B_{t,2}^2\right), \quad \mathcal{V}_2 = -\frac{3B_{t,2}^2}{28\pi} \tag{29}$$

and of $\mathcal{H}$

$$\mathcal{H}_1 = -\frac{1}{8\pi}\left(B_{t,1}^2 + \frac{5}{7}B_{t,2}^2\right), \quad \mathcal{H}_2 = -\frac{9B_{t,2}^2}{56\pi}. \tag{30}$$

3.5 Turbulent velocity field

The large-scale average of the inertia of turbulent velocity $\boldsymbol{F} = -\overline{\rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}}$ is evaluated in the local Cartesian system with axes parallel to $\boldsymbol{e} = (e_r, e_\theta, e_\phi)$

(effects of the curvature are negligible for this small-scale velocity field). Then we have

$$F_i = -\overline{(\rho v_j v_i)}_{;j} + \overline{(\rho v_j)_{;j} v_i}.$$

We use the following relations

$$\overline{\rho v_j v_i} = \delta_{ji}\overline{\rho v_i^2} \quad \text{and} \quad \overline{(\rho v_j)_{;j} v_i} = -\overline{\frac{\partial \rho}{\partial t} v_i} = 0.$$

That is we assume uncorrelated velocity components. Hence, we have

$$F_i = -\overline{(\rho v_i^2)}_{;i} \quad \text{no summation over } i\ ! \tag{31}$$

Further we allow the vertical component (r) to be statistically different from the two horizontal (θ and ϕ) ones. In this, values of $\overline{\rho v_j^2}$, are treated as functions of depth, and slowly varying functions of the co-latitude, The latter dependence is represented in the form of a truncated Legendre polynomial series,

$$\overline{\rho v_i v_j} = \rho \delta_{ij} \sum_{k=0} [\delta_{jr} \mathcal{T}_k^V(r) + \frac{1}{2}\mathcal{T}_k^H(\delta_{j\theta} + \delta_{j\phi})] P_{2k}(\cos\theta), \tag{32}$$

where we included only terms that are symmetric about equator.

Inserting this expression in Eq. (31) and using definition (14), we get

$$\mathcal{V}_k = \rho \mathcal{T}_k^V, \qquad \text{and} \qquad \mathcal{H}_k = \rho \frac{1}{2}\mathcal{T}_k^H. \tag{33}$$

3.6 Small-scale random magnetic field

Our treatment of the small-scale magnetic field is analogous to that of turbulent velocities. That is, the correlation matrix for the field components is represented in the form of the following Legendre polynomial series,

$$\overline{B_i B_j} = \delta_{ij} \sum_{k=0} [\delta_{jr} \mathcal{M}_k^V(r) + \frac{1}{2}\mathcal{M}_k^H(\delta_{j\theta} + \delta_{j\phi})] P_{2k}(\cos\theta). \tag{34}$$

Components of the Lorentz force, treated locally as Cartesian, are given by

$$F_i = \frac{1}{4\pi} \sum_{j \neq i} \left(B_j B_{i;j} - \frac{1}{2}(B_j^2);i \right),$$

Averaging over wide zonal areas and making use of $B_{i;i} = 0$, we get

$$\overline{B_\theta B_{r;\theta} + B_\phi B_{r;\phi}} = -\overline{B_r\left(B_{\theta;\theta} + B_{\phi;\phi}\right)} = \frac{1}{2}\frac{\partial \overline{B_r^2}}{\partial r}.$$

Thus

$$F_r = \frac{1}{8\pi}\frac{\partial}{\partial r}\left(\overline{B_r^2} - \overline{B_H^2}\right),$$

which is the same expression as obtained by GETAL. To evaluate the horizontal force, we use

$$\overline{B_\phi B_{\theta;\phi}} = -\overline{B_{\phi;\phi} B_\theta} = \frac{1}{2}(\overline{B_\theta^2})_{;\theta} + \overline{B_{r;r} B_\theta},$$

to get

$$F_\theta = \frac{1}{4\pi}\left[\overline{(B_r B_\theta)}_{;r} + \frac{1}{2}\left(\overline{B_\theta^2} - \overline{B_\phi^2} - \overline{B_r^2}\right)_{;\theta}\right] = -\frac{\overline{(B_r^2)}_{;\theta}}{8\pi}.$$

Finally, for the coefficients in the expansion of the magnetic pressure, we get

$$\mathcal{V}_k = \frac{\mathcal{M}_k^H - \mathcal{M}_k^V}{8\pi} \quad \text{and} \quad \mathcal{H}_k = \frac{\mathcal{M}_k^V}{8\pi}. \tag{35}$$

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4 Perturbed eigenfrequencies

Since we want to consider terms which are quadratic in velocity, in principle we need to consider a second order perturbational expression, which is

$$\Delta\omega = -\frac{C}{2I} + \frac{1}{2I\omega}\left(\frac{C^2}{4I} - \omega\Delta C + \Delta D_s + D_v + D_M\right), \tag{36}$$

where

$$\Delta D_s = \Delta(D_p + D_g) - \omega^2 \Delta I.$$

Actually we will not calculate the C integral and its perturbation but only comment on the role of the terms C for various velocity fields. The D_s integral may be calculated considering both Eulerian and Lagrangian perturbations. The result must be the same. The D_v and D_M integrals are treated as perturbations.

4.1 Data on solar frequency perturbations

Solar frequency data are given in the form

$$\nu_{\ell,n,m} \equiv \frac{\omega}{2\pi} = \bar{\nu}_{\ell,n} + \sum_{k=1} a_k \mathcal{P}_k^\ell(m), \tag{37}$$

where the $\mathcal{P}$ are orthogonal polynomials (see Ritzwoller & Lavely 1991 and Schou et al. 1994). The remaining symbols in this equation have their usual meanings. This representation ensures that the $\bar{\nu}_{\ell,n}$ are a probe of the spherical structure while the a_{2k} – the even-a coefficients – are a probe of the symmetrical (about the equator) part of distortion described by corresponding $P_{2k}(\cos\theta)$ Legendre polynomials.

We have

$$Q_{k,\ell}^m \equiv \int_0^{2\pi}\int_{-1}^{1} |Y_\ell^m|^2 P_{2k} d\mu d\phi = S_{k,\ell}\mathcal{P}_{2k}^\ell(m) \tag{38}$$

where $\mu = \cos\theta$ and

$$S_{k,\ell} = (-1)^k \frac{(2k-1)!!}{k!} \frac{(2\ell+1)!!}{(2\ell+2k+1)!!} \frac{(\ell-1)!}{(\ell-k)!}.$$

We identify here $\Delta\omega$ the difference between a solar frequencies varying in the activity cycle and frequencies calculated for spherical and static model of the neglecting effects of the magnetic and velocity field. For the m-independent part we use for $\Delta\omega$ the difference between the current and the activity minimum frequencies.

To describe frequency differences of individual modes, it is convenient to introduce quantities $\gamma_{k,\ell,n}$ through the following two relations,

$$\Delta\bar{\nu}_{\ell,n} = \frac{\gamma_{0,\ell,n}}{\tilde{I}_{\ell,n}} \tag{39}$$

and

$$a_{2k,\ell,n} = S_{k,\ell} \frac{\gamma_{k,\ell,n}}{\tilde{I}_{\ell,n}}, \tag{40}$$

where

$$\tilde{I}_{\ell,n} = \frac{I_{\ell,n}}{\sqrt{\bar{\rho}_\odot R_\odot^5}}. \tag{41}$$

is the dimensionless mode inertia calculated for the reference model with

$$y_{\ell,n}(r_{\rm phot}) = 2 \times 10^4$$

being adopted normalization of the eigenfunctions. With this normalization, the maximum values of the γ's are in the $(0.1 - 1)\mu$Hz range. Data show that the $1/I$ factor takes care of most of the mode-dependence in $\Delta\bar{\nu}$ and the even-a coefficients. In all subsequent expressions, the ℓ, n subscripts and the m superscript will not be given unless necessary.

4.2 Calculations of D_s for centroid frequency shift

Here the Lagrangian perturbations are more convenient. Since we have $\delta(drr^2\rho) = 0$, there is no contribution from ΔI. Furthermore, with our approximation for the eigenfunctions, the contribution from D_g is negligible. From Eqs.(6) with use of (9) we get

$$\Delta D_{s,0} \approx \Delta D_p = \int drr^2\rho \left[\delta(c^2)\lambda^2 + 2\delta\left(\frac{p}{\rho}\right)\Lambda(\mathcal{E} + z\lambda) + 2\frac{p}{\rho}(\Gamma\lambda + \Lambda z)\delta\lambda\right],$$

where

$$\mathcal{E} = y^2 + \Lambda z^2$$

Note that we have

$$\tilde{I} = \int dx x^4 \frac{\rho}{\bar{\rho}}\mathcal{E}, \tag{42}$$

where $x = r/R$.

With the approximation adopted in Eqs.(7) and (18), we have

$$\delta\lambda \approx -\frac{d\delta r}{dr}\frac{dy}{dr} \approx \frac{\delta\rho}{\rho}(\lambda + \Lambda z).$$

Thus, with the use of (19), we obtain

$$\Delta D_{s,0} = \int drr^2(\mathcal{D}_{s,0}\delta\mathcal{V}_0 + p\frac{\chi_T}{\chi_\rho}\mathcal{D}_T\frac{\delta T}{T}), \tag{43}$$

where

$$\mathcal{D}_{s,0} = -\Gamma\left(1 + \Gamma_p + \frac{1+\Gamma_\rho}{\chi_\rho}\right)\lambda^2 + \frac{2\Lambda}{\chi_\rho}\left[(1-\chi_\rho)\mathcal{E} - (\Gamma+\chi_\rho)z\lambda - \Lambda z^2\right] \tag{44}$$

and

$$\mathcal{D}_T = -\Gamma(1+\Gamma_\rho)\lambda^2 + 2\Lambda(\mathcal{E} - \Gamma z\lambda - \Lambda z^2) \tag{45}$$

where we denoted by Γ_p and Γ_ρ logarithmic derivatives of Γ.

If instead of (18) we use (19), then we we get

$$\Delta D_{s,0} = \int drr^2(\mathcal{D}_{s,\mathrm{ad},0}\delta\mathcal{V}_0 + p\frac{\chi_T}{\chi_\rho}\mathcal{D}_T\frac{\delta S}{c_p}), \tag{43a}$$

where

$$\mathcal{D}_{s,\mathrm{ad},0} = \mathcal{D}_{s,0} - \nabla_{\mathrm{ad}}\frac{\chi_T}{\chi_\rho}\mathcal{D}_T = -[\Gamma(1+\Gamma_p)+1+\Gamma_\rho]\lambda^2 - 2\Lambda\left[\left(1-\frac{1}{\Gamma}\right)\mathcal{E} + 2z\lambda + \frac{\Lambda z^2}{\Gamma}\right]. \tag{44a}$$

4.3 Calculations of the D_s for the splittings

In the present application it is more convenient to consider Eulerian perturbations of the structural parameters. Consider distortion proportional to P_k. We will see that within our approximation all the angular integrals appearing in ΔI, ΔD_p and δD_g reduce to Q_k. These factors take care of the k and m dependence. The property is self-evident in the case of D_g. From (10) we have

$$\Delta D_g = -2Q_k\int drr^3 g\rho_k y^2$$

and with (21) after one integration by parts and use of (7) we get

$$\Delta D_g = 4Q_k\int drr^2(\mathcal{V}_k - \mathcal{H}_k)y(\lambda + \Lambda z). \tag{46}$$

The cases of and I_p and D_p are more involved. We first note that

$$\int_1^1 d(\cos\theta)\int_0^{2\pi} d\phi|\boldsymbol{\nabla}Y_\ell^m|^2 = [\Lambda - k(2k+1)]Q_k \approx \Lambda Q_k \quad \text{if} \quad \ell \gg k. \tag{47}$$

Last approximation is not valid for low degree modes but the term involving this factor are significant for such modes only in deep layers, which we assume unperturbed. Thus, for ΔI we have approximately

$$\Delta I = Q_k \int drr^4 \rho_k \mathcal{E}.$$

Again we make use of (21) and integrate by parts and with (7) and (8), we obtain approximately

$$\Delta I = 2Q_k \int dr \frac{r^3}{g}(\mathcal{V}_k - \mathcal{H}_k)(y\lambda + 2\Lambda zy). \tag{48}$$

In calculating ΔD_p we first note that the $[...]_\phi$ term in (9) does not contribute, which follows from the assumed axial symmetry of perturbation. The contribution from the $[...]_\theta$ is nonzero, but it is small as may be justified as follows. Integrating by parts over θ one gets the $k(k+1)Q_k$ factor from the angular integral and the whole contribution from this term is of the same order as the one neglected above. Thus, we have

$$\Delta D_p = Q_k \int drr^2 \left[\Gamma\left(p_k(1+\Gamma_p) + \rho_k \frac{p\Gamma_\rho}{\rho}\right)\lambda^2 + p_k 2\Lambda(\mathcal{E} + z\lambda)\right]. \tag{49}$$

The quantity ρ_k is eliminated again by integration by parts. Here we use

$$\frac{d}{dr}\left(\frac{p\Gamma}{g\rho}\lambda^2\right) \approx \left(2 + \Gamma\frac{d\ln c^2}{d\ln p}\right)\lambda^2 + 2\Gamma\lambda\left(\Lambda z - \frac{\omega^2 r}{g}y\right),$$

which follows from Eqs.(7) and (8). The result is

$$\Delta D_p = -Q_k \int drr^2 \{\mathcal{H}_k[\Gamma(1+\Gamma_p)\lambda^2 + 2\Lambda(\mathcal{E} + z\lambda)] + (\mathcal{H}_k - \mathcal{V}_k)\Gamma_\rho \psi\},$$

where

$$\psi \equiv \left(2 + \Gamma\frac{d\ln c^2}{d\ln p}\right)\lambda^2 + 2\Gamma\lambda\left(\Lambda z - \frac{\omega^2 r}{g}y\right) + \frac{d\ln\Gamma_\rho}{d\ln p}\lambda^2.$$

Combining this with Eq.(48) in (41) (ΔD_g is negligible) we obtain

$$\Delta D_{s,k} = Q_k \int drr^2 (\mathcal{D}_s^V \mathcal{V}_k + \mathcal{D}_s^H \mathcal{H}_k), \tag{50}$$

where we denoted

$$\mathcal{D}_s^V \equiv -\frac{2\omega^2 r}{g}(y\lambda + 2\Lambda zy) + \Gamma_\rho \psi \tag{51}$$

and

$$\mathcal{D}_s^H = \frac{2\omega^2 r}{g}(y\lambda + 2\Lambda zy) - \Gamma_\rho\psi - \Gamma(1+\Gamma_p)\lambda^2 - 2\Lambda(\mathcal{E} + z\lambda). \tag{52}$$

5 Frequency perturbation due to steady flows

Linear effects are source of odd-a coefficients and there is evidence for changes during this activity cycle, but this is outside the scope of the discussion here.

It is easy to see that the linear term due to meridional circulation vanishes. First, note that we have

$$\boldsymbol{\xi}^* \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} = \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{\nabla})|\boldsymbol{\xi}|^2.$$

Then we use it in Eq.(3) and with the Gauss theorem get, upon ignoring the surface integral,

$$C = 2\mathrm{i} \int d^3\boldsymbol{x} \mathrm{div}(\boldsymbol{v}\rho) = 0.$$

Quadratic effects of rotation were discussed in detail (DG, 1992. The only significant though secondary contributor to even-a coefficient is that due to centrifugal distortion through the D_s term. The estimate of the resulting frequency shift is

$$\left(\frac{\Delta\omega_\ell^m}{\omega}\right)_{\mathrm{cen}} = \frac{\Lambda - 3m^2}{4\Lambda - 3}\frac{\epsilon_{\mathrm{cent}}}{I}\int drr^4\rho\left(\frac{r}{R}\right)^3\left(y^2 + \frac{\Lambda}{4}z^2\right), \tag{53}$$

where

$$\epsilon_{\mathrm{cent}} = \frac{\Omega^2}{\pi G\bar{\rho}} \approx 3 \times 10^{-5}$$

In our analysis of the even-a' data the constant contribution to a_2 is removed.

The second order effects of the meridional circulation do not vanish but they are by about an order of magnitude less than those due to rotation. In the estimate we ignore the quadratic effects of the Coriolis force. We note that again that the dominant contribution to $\delta\omega$ arises from D_s. Further we note that $|\mathcal{H}_{\mathrm{mc}}| \sim \rho h^2 \gg |\mathcal{V}_{\mathrm{mc}}|$. With this, ignoring derivatives of Γ we get

$$\left(\frac{\Delta\omega}{\omega}\right)_{\mathrm{mc}} = \frac{Q_k}{I}\int dr\frac{r^3\mathcal{H}_k}{g}\left[y\lambda + 2\Lambda zy - \frac{g[\Gamma\lambda^2 + 2\Lambda(\mathcal{E} + z\lambda)]}{\omega^2 r}\right] \tag{54}$$

Now compare this to with the contribution to $\delta\omega$ from the centrifugal distortion given in Eq.(53), focusing on the outermost layers. We have there $v_{\mathrm{mc}} \sim 20$ m/s and $v_{\mathrm{rot}} \sim 2$ km/s. With these numbers

$$\left|\frac{\mathcal{H}_k}{\rho g r \epsilon}\right| \sim \left(\frac{v_{\mathrm{mc}}}{v_{\mathrm{rot}}}\right)^2 \sim 10^{-4}$$

In these layers, we have $|\lambda/y| \sim 10^3$, which implies that the contribution from the meridional to $\Delta\omega$ is indeed by an order of magnitude less than that from the centrifugal force, hence may be neglected.

6 Frequency perturbation due the quadrupolar toroidal field

We assume the toroidal field in the form given in Eq.(28) but select only the $j = 2$ term. The consecutive terms in Eq.(12) are calculated in the same approximation as used in sections 2 and 3.5. Then, we get

$$|(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}|^2 = \frac{15}{4}B_t^2 m^2\mu^2(y^2|Y_\ell^m|^2 + z^2|\boldsymbol{\nabla}_H Y_\ell^m|^2),$$

$$-2\mathrm{div}\boldsymbol{\xi}^*\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\xi} = -\frac{15}{2}B_t^2 m^2\mu^2|Y_\ell^m|^2 z\lambda,$$

and

$$\frac{1}{2}|\boldsymbol{B}|^2(\Xi + |\mathrm{div}\boldsymbol{\xi}|^2) = \frac{15}{4}B_t^2(\mu^2-\mu^4)|Y_\ell^m|^2[\lambda^2 + \Lambda(\mathcal{E}+z\lambda)].$$

Consider now the surface integrals. They all may be expressed in terms of the Q_k coefficients with $k \le 2$. In the last integral we use

$$\mu^2 - \mu^4 = \frac{2}{15} + \frac{2}{21}P_2 - \frac{8}{35}P_4.$$

For calculating surface integrals appearing in the former two terms, we rely on the following recursion formula

$$q_k = \frac{2k-1}{k(4\Lambda+1-4k^2)}\{q_{k-1}[2\Lambda-(2k-1)^2-2m^2] + q_{k-2}(2k-3)(k-1)\},$$

where

$$q_k = \int_{-1}^{1}d\mu\int_0^{2\pi}d\phi\mu^{2k}|Y_\ell^m|^2,$$

to get

$$m^2\int_{-1}^{1}d\mu\int_0^{2\pi}d\phi\mu^2|Y_\ell^m|^2 = \frac{1}{2}+\frac{2\Lambda-9}{2}q_1-\frac{4\Lambda-15}{3}q_2 = \frac{\Lambda}{15}-\frac{2\Lambda+1}{21}Q_1-\frac{32\Lambda-120}{105}Q_2$$

Finally, we have

$$D_M = \frac{1}{112\pi}\int drr^2B_t^2\{(\mathcal{E}-2z\lambda)[7\Lambda-(10\Lambda+15)Q_1-(32\Lambda-120)Q_2]+$$
$$+[\lambda^2+\Lambda(\mathcal{E}+\lambda z)](14+10Q_1-24Q_2)\},$$

where $\mathcal{E} = y^2 + \Lambda z^2$.

Full expression for the frequency shift the toroidal magnetic field of the adopted form is obtain after adding the effect of the induced structural changes. The expression is

$$(\Delta\omega)_t = \frac{1}{8\pi I\omega}\int drr^2B_t^2(\mathcal{R}_{t,0}+\mathcal{R}_{t,1}Q_1+\mathcal{R}_{t,2}Q_2), \qquad (55)$$

where

$$\mathcal{R}_{t,0} = \frac{1}{4}(\mathcal{D}_{s,0,\mathrm{ad}} + 2\lambda^2 + 3\Lambda\mathcal{E}),$$

$$\mathcal{R}_{t,1} = \frac{5}{28}[\mathcal{D}_s^V - 2\mathcal{D}_s^H + 2\lambda^2 - 3\mathcal{E} + (6\Lambda + 6)z\lambda],$$

$$\mathcal{R}_{t,2} = -\frac{1}{28}\left[6(2\mathcal{D}_s^V + 3\mathcal{D}_s^H) + 24\lambda^2 + (56\Lambda - 120)\mathcal{E} - 40(\Lambda - 6)z\lambda\right].$$

The quantities $\mathcal{D}_{s,0,\mathrm{ad}}$, $\mathcal{D}_s^V$, and $\mathcal{D}_s^H$, are given in Eqs.(44a), (51), and (52), respectively. We have chosen adiabatic approximation for the radial perturbation on the ground that the layer where the field is expected is located deep enough. For an ideal gas we have

$$\mathcal{R}_{t,0} = -\frac{1}{4}\left[(\Gamma - 1)\lambda^2 + 4\Lambda z\lambda + 2\frac{\Lambda^2}{\Gamma}z^2 - \left(1 + \frac{2}{\Gamma}\right)\Lambda\mathcal{E}\right],$$

$$\mathcal{R}_{t,1} = -\frac{5}{28}\left[\frac{6\omega^2 r}{g}y(\lambda + 2\Lambda z) - 2(\Gamma + 1)\lambda^2 + (4\Lambda - 3)\mathcal{E} + (10\Lambda + 6)z\lambda\right],$$

$$\mathcal{R}_{t,2} = \frac{1}{28}\left[-\frac{12\omega^2 r}{g}y(\lambda + 2\Lambda z) - 6(3\Gamma - 4)\lambda^2 + 20(\Lambda - 6)\mathcal{E} - 4(\Lambda - 60)z\lambda\right].$$

We now rewrite (55) in the following convenient form, specialized for the sun

$$\gamma_{t,k} = \int_0^1 dx K_{t,k}\left(\frac{B_t}{1\ \mathrm{MG}}\right)^2 \quad \mu\mathrm{Hz}, \tag{56}$$

with $x = r/R_\odot$

$$K_{t,k} = N_t\frac{1\ \mathrm{mHz}}{\nu}x^2\mathcal{R}_{t,k},$$

(for normalization of the eigenfunctions see the text after Eq.(41)) and

$$N_t = \frac{10^{9+12}}{32\pi^3\bar{\rho}_\odot R_\odot^2} = \frac{10^{21}R_\odot}{2.4\pi^2 M_\odot} = 1.48 \times 10^{-4}.$$

In Fig.1 we show kernels $K_{t,k}(x)$ in the vicinity of the bottom of the solar convective zone, where the toroidal field of the form given in Eq.(28) is believed present. . The total effect is marginally detectable if $B_t \sim 0.1$MG , which is now regarded upper limit for such a field (e.g. D'Silva & Howard, 1993).

7 Effects turbulent pressure perturbation

There is no net contribution from the linear term in v because we have

$$\overline{\rho\boldsymbol{\xi}^* \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi}} = 0.$$

However, we were unable to justify our neglect of the second term $\omega\Delta C$. We may only hope that we capture dominant effects of the random velocity field

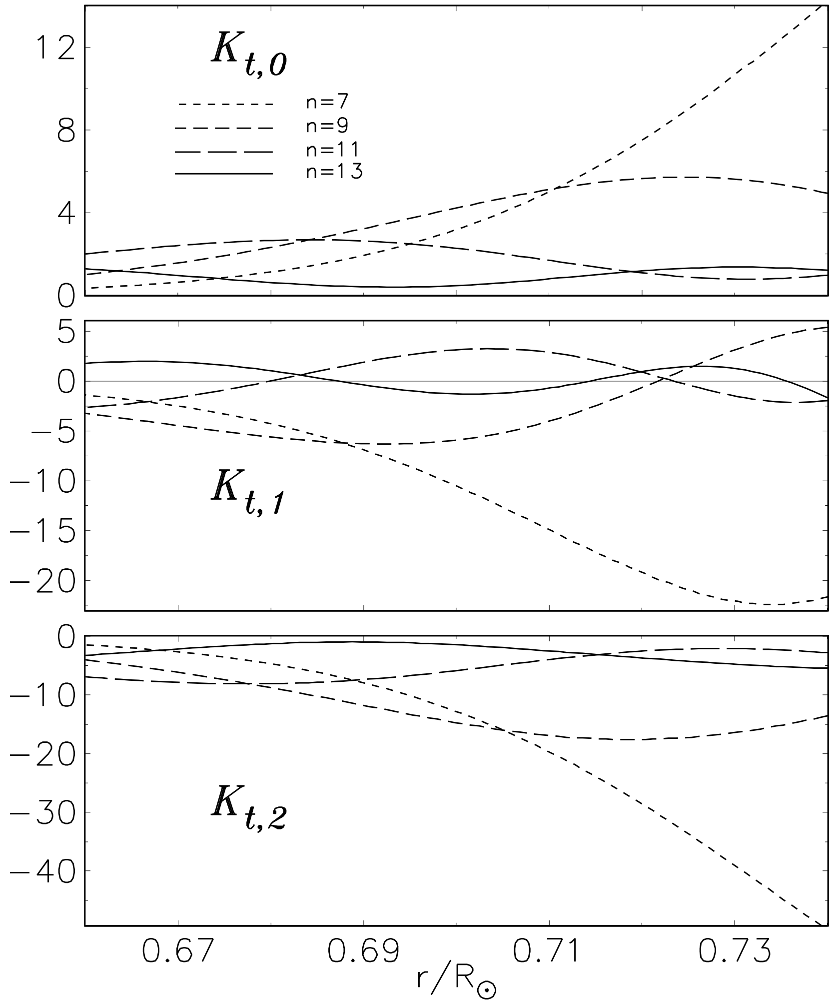

Figure 1: Kernels for calculating $\gamma_{t,k}$, according to Eq.(56) for four $\ell = 30$ modes of indicated orders n in the zone near the bottom of the convective envelope ($x = r/R_\odot = 0.71$). The modes have frequencies in the 2.1 to 3.1 mHz range. The lower turning points are range accordingly from $x = 0.72$ to 0.6

considering only D_v and ΔD_s in Eq.(41). The effects included are reduced to perturbation of an anisotropic turbulent pressure.

For evaluating D_v according to Eq.(11), we use the random velocity field representation given in Eq.(32). We note that

$$\overline{\rho|(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}|^2} = \rho\sum_k \left(\mathcal{T}_k^V A^V + \frac{1}{2}\mathcal{T}_k^H A^H\right) P_{2k}, \tag{57}$$

where

$$A^V \approx \left(r\frac{dy}{dr}\right)^2 |Y_\ell^m|^2 + \left(r\frac{dz}{dr}\right)^2 |\boldsymbol{\nabla} Y_\ell^m|^2$$

and

$$A^H \approx y^2|\boldsymbol{\nabla} Y_\ell^m|^2 + z^2[|(Y_\ell^m)_{;\theta;\theta}|^2 + 2|(Y_\ell^m)_{;\theta;\phi}|^2 + |(Y_\ell^m)_{;\phi;\phi}|^2].$$

The radial derivatives in A^V are eliminated with the approximate Eqs.(7) and (8). The angular integrals are evaluated by parts keeping only derivatives of the spherical harmonics. This approximation justifies, in particular, the replacement

$$2|(Y_\ell^m)_{;\theta;\phi}|^2 \rightarrow 2\Re[(Y_\ell^m)_{;\theta;\theta}(Y_\ell^m)_{;\phi;\phi}].$$

In this way we get for the contribution to D_v from the P_{2k} component of the random velocity field

$$D_{v,k} = -Q_k\int drr^2\rho\left[\mathcal{T}_k^V(\lambda^2 + 2\Lambda z\lambda + \Lambda\mathcal{E}) + \mathcal{T}_k^H\frac{\Lambda}{2}\mathcal{E}\right]$$

Contribution to $D_{s,0}$ from the induced changed in gas pressure pressure is given by

$$\Delta D_s = \int drr^2\rho\mathcal{D}_{s,0}\delta\mathcal{T}_0^V.$$

Adding these two expression (see Eqs. 36), we get

$$(\Delta\omega)_{v,0} = \frac{1}{2\omega I}\int drr^2\rho(\mathcal{R}_{v,0}^V\delta\mathcal{T}_0^V + \mathcal{R}_{v,0}^H\delta\mathcal{T}_0^H), \tag{58}$$

where

$$\mathcal{R}_{v,0}^V = \mathcal{D}_{s,0} - (\lambda^2 + 2\Lambda z\lambda + \Lambda\mathcal{E})$$

and

$$\mathcal{R}_{v,0}^H = -\frac{\Lambda}{2}\mathcal{E}.$$

The expression for $\mathcal{D}_{s,0}$ is given in Eq.(44)

For $k > 0$ we get from Eqs.(50) and (33)

$$\Delta D_{s,k} = Q_k\int drr^2\rho(\mathcal{D}_s^V\mathcal{T}_k^V + \frac{1}{2}\mathcal{D}_s^H\mathcal{T}_k^H).$$

This together with the expression for $D_{v,k}$ used in Eq.(36) gives

$$(\Delta\omega)_{v,k} = \frac{Q_k}{2I\omega}\int drr^2\rho(\mathcal{R}_v^V\mathcal{T}_k^V + \mathcal{R}_v^H\mathcal{T}_k^H), \tag{59}$$

where

$$\mathcal{R}_{v,k}^V = \mathcal{D}_s^V - (\lambda^2 + 2\Lambda z\lambda + \Lambda\mathcal{E})$$

and

$$\mathcal{R}_{v,k}^H = \frac{1}{2}(\mathcal{D}_s^H - \Lambda\mathcal{E}).$$

The general expressions for $\mathcal{D}_s^V$ and $\mathcal{D}_s^H$ are given in Eqs.(51) and (52), respectively. Note that $\mathcal{R}_{v,k}^V$ and $\mathcal{R}_{v,k}^H$ are the same for all $k > 0$. Now we can use our result given in Eqs.(58) and (59) to evaluate γ's introduced in Section 4.1. We get

$$\gamma_{v,k} = \int d\left(\frac{d_{\rm phot}}{1 \text{ Mm}}\right)\left[K_{v,k}^V\left(\frac{\delta\mathcal{T}_k^V}{1 \text{ km}^2\text{s}^{-2}}\right) + K_{v,k}^H\left(\frac{\delta\mathcal{T}_k^H}{1 \text{ km}^2\text{s}^{-2}}\right)\right] \quad \mu \text{ Hz}, \tag{60}$$

where $d_{\rm phot}$ is the depth beneath the photosphere,

$$K_{v,k}^{V,H} = N_v\frac{1 \text{ mHz}}{\nu}\frac{\rho}{\bar{\rho}}x^2\mathcal{R}_{v,k}^{V,H},$$

and

$$N_v = \frac{10^{6+3+8+10}}{8\pi^2(R_\odot)^3} = 3.76\times10^{-8}$$

In Fig.2 we show three of the K_v kernels that are important for evaluating γ's for p-modes due activity related perturbation of the turbulent pressure. The kernels for multiplying the vertical component, have significantly larger absolute values and are negative in these layers. With the help of the figure we may roughly estimate required change in in the velocity fluctuations needed to account for the measured γ's on the assumption that this change is their only source. From numerical simulations we know (e.g. Abbett et al., 1997) that velocity fluctutions at the level of 1 km/s persist in the whole layer shown with the maximum of nearly 3 km/s at $d_{phot} \approx 0.1$. The frequency averaged, value of $\gamma_0 \approx 0.3\mu Hz$ (Dziembowski et al., 2001) requires some a few percent change in the radial component of velocity fluctuations. This seems not much. A smilar requirements are found to account for measured γ_k at $k > 0$.

Fig 3. shows that the kernels for the f-modes are very different. An estimate shows, however, also for f-modes the changes in turbulent pressure most be regarded as potentially significant source of the frequency changes.

More detailed analysis of the $\gamma_k(\nu)$ dependence reflecting the kernels frequency dependence seen in Fig. 2 is needed to say more about the nature of the required change in the convective velocities. However, already at this stage we may conclude that this change must be regarded as important, perhaps the dominant, contributor to solar p-mode frequency changes in the activity cycle and the sign of the observed changes agrees with the expected inhibiting effect of the field on convective velocities.

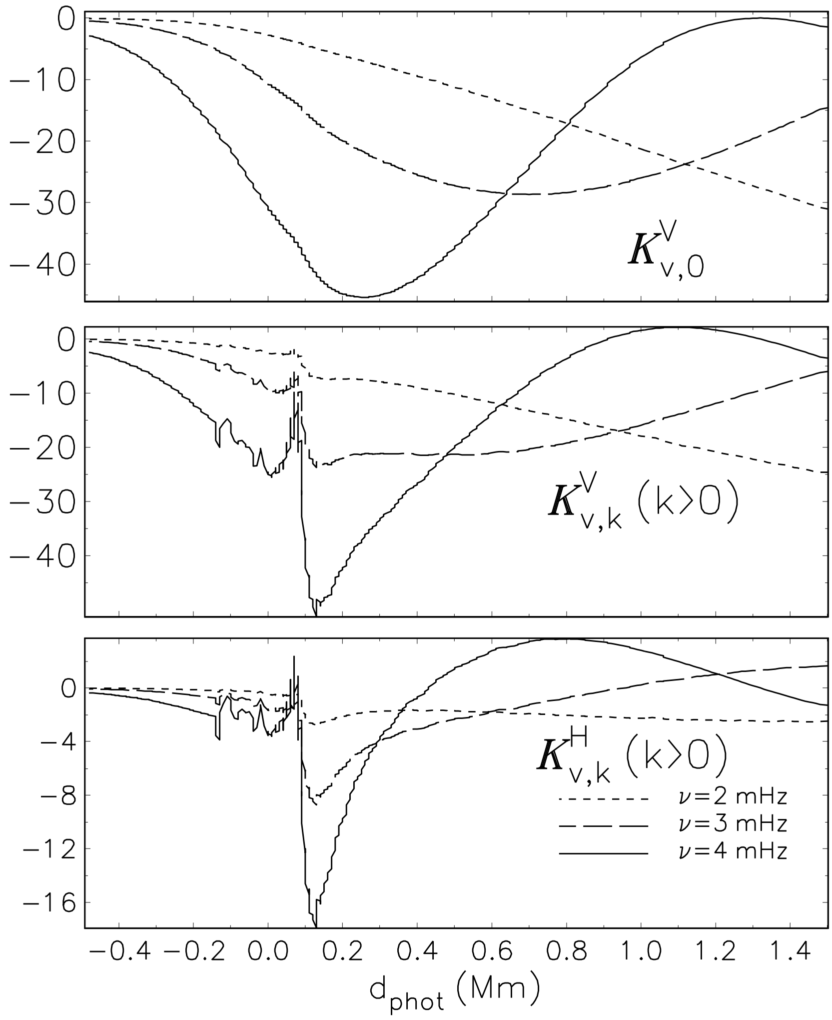

Figure 2: Kernels for calculating $\gamma_{v,k}$ according to Eq.(60) at three selected frequencies plotted as functions of the depth in outer part of the standard solar models. The three kernels are ℓ-indpendent for p-modes in this part of the sun. The $K^H_{v,0}$ kernel is $\propto \Lambda$ and much smaller.

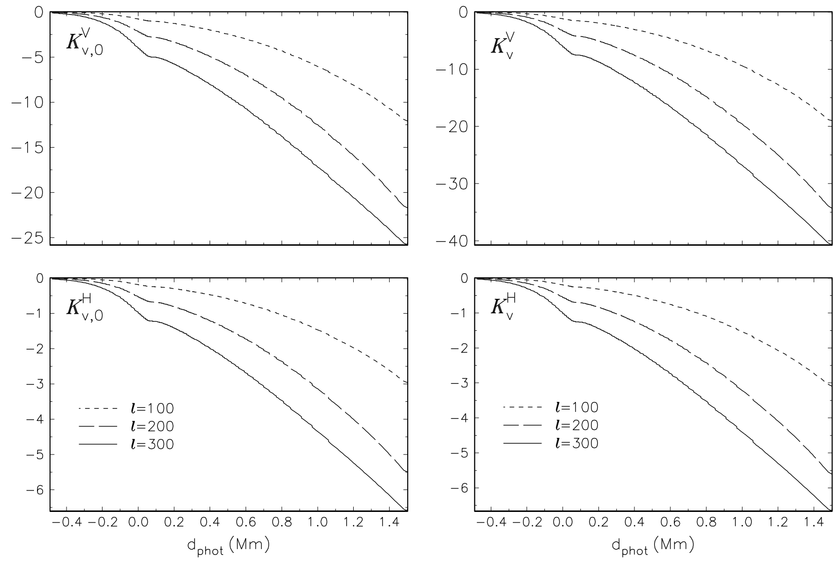

Figure 3: Kernels for calculating $\gamma_{v,k}$ for f-modes at indicated three ℓ degrees.

8 Effect of a small scale magnetic field

With the random field described by a single P_k-component (see Eq.(34)), the three terms in the integrand of Eq.(12) are transformed as follows

$$\overline{|(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}|^2} \to \left[\mathcal{M}^V_k(\lambda^2+2\Lambda z\lambda+\Lambda\mathcal{E})+\mathcal{M}^H_k\frac{\Lambda}{2}\mathcal{E}\right]|Y^m_\ell|^2,$$

$$-2\overline{\mathrm{div}\boldsymbol{\xi}^*\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}} \to -[2\mathcal{M}^V_k\lambda(\lambda+\Lambda z)-\mathcal{M}^H_k\Lambda z\lambda]|Y^m_\ell|^2,$$

and

$$\frac{1}{2}\overline{|\boldsymbol{B}|^2}(\Xi+|\mathrm{div}\boldsymbol{\xi}|^2)\to(\mathcal{M}^V_k+\mathcal{M}^H_k)[\lambda^2+\Lambda(z\lambda+\mathcal{E})]|Y^m_\ell|^2.$$

The transformations use integration by parts over the surface, our approximations regarding the eigenfunctions, and the angular dependence of $\mathcal{M}$. Note that the first term is fully analogous to the integrand in D_v considered in the previous section and the third to one appearing in the integrand of D_p.

With the above expression we get from Eq.(12)

$$D_{M,k}=\frac{Q_k}{4\pi}\int drr^2\{\delta\mathcal{M}^V_k\Lambda(z\lambda+2\mathcal{E})+\delta\mathcal{M}^H_k[\lambda^2+\Lambda(2z\lambda+1.5\mathcal{E})]\}. \quad (61)$$

For spherically symmetric part of ΔD_s we use the isothermal expression given in Eq.(43), which combined with Eq.(35), yields

$$\Delta D_{s,0} = \frac{1}{8\pi} \int drr^2 \mathcal{D}_{s,0} \delta(\mathcal{M}_0^H - \mathcal{M}_0^V). \tag{62}$$

Using last two expressions in eq.(36), w get

$$(\Delta\omega)_{M,0} = \frac{1}{16\pi\omega I} \int drr^2 (\mathcal{R}^V_{M,0} \delta \mathcal{M}_0^V + \mathcal{R}^V_{M,0} \delta \mathcal{M}_0^H), \tag{63}$$

where

$$\mathcal{R}^V_{M,0} = 2\Lambda(z\lambda + 2\mathcal{E}) - \mathcal{D}_{s,0}$$

and

$$\mathcal{R}^H_{M,0} = 2\lambda^2 + \Lambda(4z\lambda + 3\mathcal{E}) + \mathcal{D}_{s,0}.$$

The expression for $\mathcal{D}_{s,0}$ is given in Eq.(44)

Similarly, for $k > 0$ we have from Eqs.(50) and (35)

$$\Delta D_{s,k} = \frac{Q_k}{8\pi} \int drr^2 [(\mathcal{D}_s^H - \mathcal{D}_s^V)\mathcal{M}_k^V + \mathcal{D}_s^V \mathcal{M}_k^H)]$$

and

$$(\Delta\omega)_{M,k} = \frac{Q_k}{16\pi\omega I} \int drr^2 (\mathcal{R}^V_M \mathcal{M}_k^V + \mathcal{R}^H_M \mathcal{M}_k^H), \tag{64}$$

where

$$\mathcal{R}^V_{M,k} = \mathcal{D}_s^H - \mathcal{D}_s^V + 2\Lambda(z\lambda + 2\mathcal{E})$$

and

$$\mathcal{R}^H_{M,k} = \mathcal{D}_s^V + 2\lambda^2 + \Lambda(4z\lambda + 3\mathcal{E}).$$

Again kernels $\mathcal{R}_{M,k}$ are the same for all $k > 0$ The expression for $\mathcal{D}_s^V$ and $\mathcal{D}_s^H$ are given in Eqs.(51) and (52), respectively.

Now we have for γ's

$$\gamma_{M,k} = \int d\left(\frac{d_{\rm phot}}{1 \text{ Mm}}\right) \left[K^V_{M,k} \left(\frac{\delta\mathcal{M}_k^V}{1 \text{ kG}}\right)^2 + K^H_{M,k} \left(\frac{\delta\mathcal{M}_k^H}{1 \text{ kG}}\right)^2\right] \quad \mu \text{ Hz}, \tag{65}$$

where

$$K^{V,H}_{M,k} = N_M \frac{1 \text{ mHz}}{\nu} x^2 \mathcal{R}^{V,H}_{M,k},$$

and

$$N_M = \frac{10^{9+8+6}}{64\pi^3 \bar{\rho}_\odot R^3} = \frac{10^{23}}{48\pi^2 M_\odot} = 1.06 \times 10^{-13}.$$

Do not forget that the eigenfunction normalization in all $\mathcal{R}$'s must be the same in $\tilde{I}$ (see section 4.1)

In Figs. 4 and 5 we show p- and f-mode kernels, respectively, for calculating γ_M according to Eq. 65. In Fig. 4 note the strong sensitivity to the frequency which emphasizes the probing power of the $\gamma(\nu)$. Further, that the kernels imply that the radial field increase in outer layers will lead to a frequency increase, while that of the horizontal field is likely to have an opposite effect. Comparing

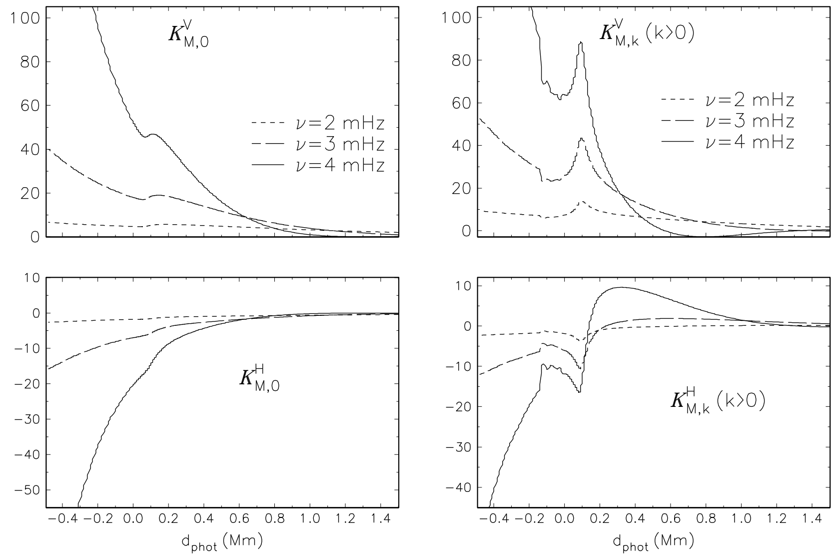

Figure 4: Kernels for calculation the p-mode γ's at three selected frequencies plotted as functions of the depth in outer part of the standard solar models.

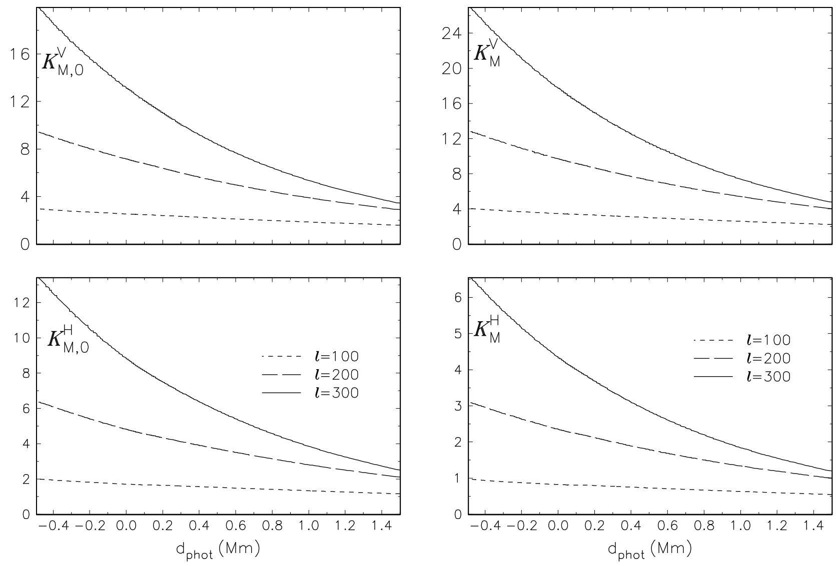

Figure 5: Kernels for calculations the f-mode γ's at three selected mode degrees plotted as functions of the depth in outer part of the standard solar models.

plots in Figs. 4 and 5, note big differences between the p- and f-mode kernels. The difference is important as it may help to disentangle the role of various contributors to the freqency changes.

Again we may use these plots to asses the required magnetic field changes needed to account for measured γ's. The minimum requirement for the field increase is obtained if we assume that only radial component increases. Only crude estimate of ~ 100 G.

9 Temperature and entropy perturbation

To evaluate centroid frequency changes For we have to add the effect of correlated temperature perturbation. Eqs. (38) (41) and (43) imply that the specialize for the sun expression is

$$(\Delta\nu)_T = \int_0^1 dx K_T \frac{\delta T}{T} \quad \mu \text{ Hz}, \tag{66}$$

where

$$K_T = 0.50 \frac{1 \text{ mHz}}{\nu} x^2 \tilde{p} \frac{\chi_t}{\chi_\rho} \frac{\mathcal{D}_T}{\tilde{I}},$$

and

$$\tilde{p} = p \frac{R}{GM\bar{\rho}}$$

is dimesionless pressure. The quantity $\mathcal{D}_T$ is given in Eq.(45). It is negative for radial modes and through most of the interior for other p-modes.

Our complete expression for the centroid frequency change due small scale magnetic field and correlated temperature change, which may be compared with that of GETAL is

$$\frac{\Delta\omega}{\omega} = \frac{1}{2\omega^2 I}(D_{M,0} + \Delta D_{s,0}).$$

We use Eq.(61) for $D_{M,0}$ and Eq.(43a) with for $\Delta D_{s,0}$to get

$$\frac{\Delta\omega}{\omega} = \frac{1}{2\omega^2 I} \int drr^2 \left(p \frac{\chi_T}{\chi_\rho} \mathcal{D}_T \frac{\delta S}{c_p} + \mathcal{R}^V_{s,0,\text{ad}} \frac{\delta \mathcal{M}_0^V}{8\pi} + \mathcal{R}^H_{s,0,\text{ad}} \frac{\delta \mathcal{M}_0^H}{8\pi} \right), \tag{67}$$

where

$$\mathcal{R}^V_{M,0,\text{ad}} = 2\Lambda(z\lambda + 2\mathcal{E}) - \mathcal{D}_{s,0,\text{ad}}$$

and

$$\mathcal{R}^H_{M,0,\text{ad}} = 2\lambda^2 + \Lambda(4z\lambda + 3\mathcal{E}) + \mathcal{D}_{s,0,\text{ad}}.$$

Kernels and $\mathcal{D}_{s,0,\text{ad}}$ and $\mathcal{D}_T$ are given in Eqs. 45 and 44a, respectively. If we set $\Lambda = 0$ everywhere we obtain

$$\frac{\Delta\omega}{\omega} = \frac{1}{2\omega^2 I} \int drr^2 \left[-p \frac{\chi_T}{\chi_\rho} \Gamma(1 + \Gamma_\rho) \frac{\delta S}{c_p} + \zeta \frac{\delta \mathcal{M}_0^V}{8\pi} + (2 - \zeta) \frac{\delta \mathcal{M}_0^H}{8\pi} \right] \lambda^2, \tag{68}$$

where

$$\zeta = \Gamma(1+\Gamma_p) + 1 + \Gamma_\rho .$$

It is easy to see that our Eq.(66) becomes GETAL's Eq.(14) if we express our parameters through theirs as follows,

$$4\pi\omega^2 I\lambda^2 = \left(\frac{\partial\xi}{\partial r}\right)^2$$

(see our Eq.(7) with $\Lambda = 0$ and their Eq.(5)),

$$\mathcal{M}_0^V = \frac{1-\beta}{2}B^2, \qquad \mathcal{M}_0^H = \frac{1+\beta}{2}B^2$$

(see their Eq.(A6) for the definition of β),

$$\frac{1}{2}\frac{\chi_T}{\chi_\rho}\frac{1+\Gamma_\rho}{c_p} = -\left(\frac{\partial\ln(\rho c)}{\partial s}\right)_p, \quad \text{and} \quad \zeta = 2\left(\frac{\partial\ln c}{\partial\ln\rho}\right)_s + 2.$$

The last two relations follows from general thermodynamical relations expressed in symbols used in this paper. These expressions may be applied to nonradial p-modes for the perturbation localized well above their lower turning points. Our Eq.(65) removes this restriction and makes expression valid also for f-modes.

For $k > 0$ the temperature perturbation is determined by the turbulent and the magnetic pressure perturbations. From Eqs.(24) (33) and (35) we get

$$\frac{\delta_k T}{T} = -\frac{1}{p\chi_T}\left[\left(\frac{1}{2}\mathcal{T}_k^H + \frac{\mathcal{M}_k^V}{8\pi}\right) + \frac{1}{2}\left(\chi_\rho\frac{d}{d\ln p} + \chi_T\frac{d\ln T}{d\ln p}\right)\left(2\mathcal{T}_k^V - \mathcal{T}_k^H + \frac{\mathcal{M}_k^H - 2\mathcal{M}_k^V}{4\pi}\right)\right]. \tag{69}$$