Electric current view of Magnetohydrodynamics

Jongchul Chae

Ampere's law The equation of the Ampere's law used in MHD

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{1}$$

tells us that MHD phenomena can be discussed in terms of either magnetic field or current. It should be remembered that this euation implies steay-state current phenomena

$$\nabla \cdot \mathbf{J} = 0 \tag{2}$$

which in turn means the absence of displacement current. In MHD, the time variations at relatively low frequencies are of interest so the charge seperation which may occur a very short time scales can be neglected. The familiar integral form of the Ampere's law

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \frac{4\pi}{c}I \tag{3}$$

can be used to determine the current passing throough a closed surface C. Note that the inverse form of Eq. 1 is given as

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} dV. \tag{4}$$

Eq. 1 regards $\mathbf{B}$ as a primary physical quantity and is useful in deriving $\mathbf{J}$ from $\mathbf{B}$. On the other hand, Eq. 4 regards $\mathbf{J}$ as the primary variable and is useful in deriving $\mathbf{B}$ when $\mathbf{J}$ is known. The physical significance of Eq. 4 is that B can not exist if $\mathbf{J} = \mathbf{0}$ everywhere. This means that even if $\mathbf{B}$ can be potential and $\mathbf{J} = \mathbf{0}$ at a certian part of the system, the existence of non-zero $\mathbf{B}$ requres the existence of $\mathbf{J}$ somewhere in the system. The existence of potential-like magnetic fields on the solar surace is even due to the current system below the surface. Eq. 4 can be re-written by introcuducting the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV \tag{5}$$

like

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{6}$$

Magnetic Energy The total magnetic energy, too, can be expressed in terms of either $\mathbf{B}$ or $\mathbf{J}$:

$$E_{\rm mag} = \int \frac{B^2}{8\pi} dV \tag{7}$$

and

$$E_{\rm mag} = \int \frac{1}{2c}\mathbf{J}\cdot\mathbf{A}dV \quad (8)$$

$$= \int\int \frac{1}{2c^2}\frac{\mathbf{J}(\mathbf{r})\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dVdV'. \quad (9)$$

If the current configuration consists of n circuits carrying currents I_i, the energy can be expressed as

$$E_{\rm mag} = \frac{1}{2}\sum_i L_i I_i^2 + \sum_i\sum_{j>i} M_{ij}I_iI_j \quad (10)$$

where the mutual inductance M_{ij} is defined by

$$M_{ij} = \frac{1}{c^2}\oint\oint\frac{d\mathbf{l}_i\cdot d\mathbf{l}_j}{|\mathbf{r}_1-\mathbf{r}_2|} \quad (11)$$

Electric Field in Astrophysical Plasma From Ohm's law and Amepere's law, it follows

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B}/c + \mathbf{J}/\sigma \quad (12)$$

$$= -\mathbf{v}\times\mathbf{B}/c + \frac{c}{4\pi\sigma}\nabla\times\mathbf{B} \quad (13)$$

where the ratio of the first to the second term in the right side is of the order of the value $4\pi(v/c)(l\sigma/c)$. In the solar corona, typical values are $v = 10$ km s^{-1}, $l = 1000$ km, and $\sigma = 10^{15}$ s^{-1} which result in 10^9 for the ratio. Thus in this kind of infinite conductive limit or large length-scale limit, we have

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B}/c. \quad (14)$$

Keep in mind that this equation may break when dealing with very small scale features at small electric conductivites. For example, slow magnetic reconnection which might occur at the temperature minimum of the Sun is characterized by $v = 0.3$ km s^{-1}, $l = 1$ km s^{-1}, and $\sigma = 10^{10}$. In this case we have the ratio of 0.4, so the second term should not be neglected.

Magnetic Energy Transport Inflow of magnetic energy produces a rise in the magnetic energy, the work done by the Lorentz force, and ohmic heat disspation:

$$-\nabla\cdot\mathbf{S} = \frac{\partial}{\partial t}\frac{B^2}{8\pi} + \frac{J^2}{\sigma} + \mathbf{v}\cdot\mathbf{J}\times\mathbf{B}. \quad (15)$$

The Poyinting vector $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \quad (16)$$

In the large scale length limit, $\mathbf{E} = -\mathbf{v}\times\mathbf{B}/c$ so that

$$\mathbf{S} = \frac{(\mathbf{v}\times\mathbf{B})\times\mathbf{B}}{4\pi} \quad (17)$$

which consists of two components

$$\mathbf{S} = -\frac{\mathbf{v} \cdot \mathbf{B}}{4\pi}\mathbf{B} + \frac{B^2}{4\pi}\mathbf{v} \tag{18}$$

where the first term is magnetic energy flow along the field lines and the second term is the magentic energy flow in the velocity direction. Note that if the velocity $\mathbf{v}$ is perpectly parallel to $\mathbf{B}$, there is no net magnetic energy transfer.